

# Instantons and Intermittency

G. Falkovich<sup>a</sup>, I. Kolokolov<sup>b</sup>, V. Lebedev<sup>a,c</sup> and A. Migdal<sup>d</sup>

<sup>a</sup> *Physics Department, Weizmann Institute of Science, Rehovot 76100, Israel*

<sup>b</sup> *Budker Institute of Nuclear Physics, Novosibirsk 630090, Russia*

<sup>c</sup> *Landau Inst. for Theor. Physics, Moscow, Kosygina 2, 117940, Russia*

<sup>d</sup> *Physics Department, Princeton University, Princeton NJ 08544-1000 USA*

We propose the new method for finding the non-Gaussian tails of probability distribution function (PDF) for solutions of a stochastic differential equation, such as convection equation for a passive scalar, random driven Navier-Stokes equation etc. Existence of such tails is generally regarded as a manifestation of intermittency phenomenon. Our formalism is based on the WKB approximation in the functional integral for the conditional probability of large fluctuation. We argue that the main contribution to the functional integral is given by a coupled field-force configuration – *instanton*. As an example, we examine the correlation functions of the passive scalar  $u$  advected by a large-scale velocity field  $\delta$ -correlated in time. We find the instanton determining the tails of the generating functional and show that it is different from the instanton that determines the probability distribution function of high powers of  $u$ . We discuss the simplest instantons for the Navier-Stokes equation.

PACS numbers 47.10.+g, 47.27.-i, 05.40.+j

## I. INTRODUCTION

The intermittency phenomenon (reflected in non-Gaussian, scaling-violating tails of PDF) is believed to be the hardest part of the yet to be built theory of turbulence. It is related to rare fluctuations which cannot be treated in terms of a perturbation theory. Neither the physical mechanism nor the mathematical properties of such fluctuations are known.

Now, what is the most likely force which can lead to the given rare fluctuation of the field? The main idea of this paper is that such force is not random at all. It satisfies the well defined equation, which follows from the WKB approximation in the functional integral. Asymptotically, the fluctuations of the force around this most likely one are negligible. In this respect, the method is similar to the “optimal fluctuation” method used at treating properties of a solid with quenched disorder (see e.g. the book [1]) or at treating high-order terms of the perturbation series in the quantum field theory [2].

The problem under consideration is quite general, it can be formulated for any field governed by a nonlinear dynamic equation and driven by a random “force”. Generally, the PDF of the field depends both on the statistics of the driven force and on the form of the dynamical equation. Here, we are interested in the second dependence so that we assume the force to be Gaussian. Because of nonlinearity, the PDF of the field is non-Gaussian even for a Gaussian random force. Note that strong intermittency appears also for linear problems with “multiplicative noise”, for instance, for a passive scalar advected by a random velocity field.

We start with the dynamical equation

$$\partial_t u + \mathcal{L}(u) = \phi, \quad (1.1)$$

that controls the evolution of a field  $u(t, \mathbf{r})$  under the action of a random “force”  $\phi(t, \mathbf{r})$ . Here  $\mathcal{L}(u)$  is a nonlinear expression, it can be thought to be local in space. Generally, both the field  $u$  and the force  $\phi$  have a number of components. The Gaussian statistics of the force  $\phi$  is completely characterized by the pair correlation function

$$\langle \phi(t_1, \mathbf{r}_1) \phi(t_2, \mathbf{r}_2) \rangle = \Xi(t_1 - t_2, \mathbf{r}_1 - \mathbf{r}_2). \quad (1.2)$$

In principle, the relations (1.1,1.2) contain all the information about the statistics of  $u$ .

The equation (1.1) describes e.g. thermal fluctuations in hydrodynamics where it is reduced to the well known Langevin equation [3]. Then  $\phi$  is short-correlated in time and in space that is it can be treated as a white noise. For some systems, this thermal noise produces remarkable dynamical effects. Some examples are collected in the book [4]. Here, we will be interested in turbulence where  $\phi$  is an external “force” correlated on large scales in space. Turbulence was first treated in terms of the equation (1.1) by Wyld [7] who formulated the diagram technique as a perturbation series with respect to the nonlinear term in the Navier-Stokes equation. The diagram technique cannot be applied to our problem since we are interested in non-perturbative effects. Nevertheless we can use the functional

that generates the technique since it is a non-perturbative object. Such generating functional was introduced in [8,9], for the equation (1.1) it has the form

$$\begin{aligned}\mathcal{Z}(\lambda) &\equiv \left\langle \exp \left( i \int dt d\mathbf{r} \lambda u \right) \right\rangle \\ &= \int \mathcal{D}u \mathcal{D}p \exp \left( i\mathcal{I} + i \int dt d\mathbf{r} \lambda u \right),\end{aligned}\tag{1.3}$$

where  $p$  is an auxiliary field and the effective action is

$$\begin{aligned}\mathcal{I} &= \int dt d\mathbf{r} p [\partial_t u + \mathcal{L}(u)] \\ &+ \frac{i}{2} \int dt dt' d\mathbf{r} d\mathbf{r}' \Xi(t-t', \mathbf{r}-\mathbf{r}') pp'.\end{aligned}\tag{1.4}$$

The coefficients of the expansion of  $\mathcal{Z}$  in  $\lambda$  are the correlation functions of  $u$ . The auxiliary field  $p$  determines response functions of the system, for instance, the linear response function (Green function) is  $G = \langle up \rangle$ . Note the remarkable property [10]

$$\int \mathcal{D}u \mathcal{D}p \exp(i\mathcal{I}) = 1,$$

related to causality. That is the reason why the normalization constant is unity in (1.3). This makes it possible to average directly  $\mathcal{Z}$  over any additional random field if necessary.

The asymptotics of  $\mathcal{Z}(\lambda)$  at large  $\lambda$  is determined by the saddle-point configuration (usually called classical trajectory or instanton) which should satisfy the following equations obtained by varying the argument of the exponent in (1.3) with respect to  $u$  and  $p$

$$\partial_t u + \mathcal{L}(u) = -i \int dt' d\mathbf{r}' \Xi(t-t', \mathbf{r}-\mathbf{r}') p(t', \mathbf{r}'),\tag{1.5}$$

$$\partial_t p - \frac{\delta \mathcal{L}}{\delta u} p = \lambda.\tag{1.6}$$

Solutions of the equations (1.5,1.6) are generally smooth functions of  $t$  and  $\mathbf{r}$ . Comparing the equations (1.1) and (1.5) we conclude that the right hand side of the equation (1.5) just describes a special force configuration necessary to produce the instanton. If  $u_{\text{inst}}$  is a solution of (1.5,1.6) then asymptotically at large  $\lambda$

$$\delta \ln \mathcal{Z}(\lambda) / \delta \lambda = i u_{\text{inst}}.\tag{1.7}$$

Let us discuss the boundary conditions for the saddle-point equations. The equation (1.5) implies that we should fix the value  $u_{\text{in}}$  for the field  $u$  at the initial time  $t_{\text{in}}$ . Contrary, a boundary condition for the field  $p$  is implied at the remote future since, as follows from (1.6), it propagates in the negative direction in time. Minimization of the action generally requires  $p \rightarrow 0$  at  $t \rightarrow \infty$ . For the instantons discussed below, the finiteness of the action will also require  $u \rightarrow 0$  at  $t \rightarrow -\infty$ .

If one is interested in the simultaneous statistics of  $u$  then the function  $\lambda$  can be chosen as

$$\lambda(t, \mathbf{r}) = y \delta(t) \lambda_0(\mathbf{r}),\tag{1.8}$$

where  $y$  is a number and  $\lambda_0$  is an appropriate function of  $\mathbf{r}$  depending on what spatial correlation functions we are going to study. In this case, we should find the solution for  $p$  satisfying the rule:  $p = 0$  at  $t > 0$ . The system (1.5,1.6) is thus to be treated for  $t < 0$  only. That corresponds to the causality principle since only processes occurring in the past could influence the value of the simultaneous correlation functions at  $t = 0$ . The formal ground for the rule follows from the consideration of the problem in the restricted time interval  $t < t_0$  what is possible if  $\lambda = 0$  at  $t > t_0$ . Then the minimization of  $\mathcal{I} + \int dt d\mathbf{r} \lambda u$  over the final value  $u(t_0)$  gives  $p(t_0) = 0$  because of the boundary term originating from  $\int dt d\mathbf{r} p \partial_t u$ .

One may be interested also in the probability distribution function  $\mathcal{P}(u)$  for the field  $u$ . It can be expressed via the generating functional  $\mathcal{Z}(\lambda)$  via the functional Fourier transform

$$\mathcal{P}(u) = \int \mathcal{D}\lambda \mathcal{Z}(\lambda) \exp \left( -i \int dt d\mathbf{r} \lambda u \right).\tag{1.9}$$

We expect that the behavior of  $\mathcal{P}(u)$  at large  $u$  as well as the behavior of  $\mathcal{Z}(\lambda)$  for large  $\lambda$  is associated with some saddle-point configurations. Generally, the configurations are not always the same for both (1.3) and (1.9). Indeed, we see from (1.9) that the tail of  $\mathcal{P}(u)$  at large  $u$  corresponds to a large value of  $\delta \ln \mathcal{Z}(\lambda)/\delta \lambda$  which is related to large  $\lambda$  only if the tails of both PDF and the generating functional decay faster than exponent – see the example in Sect. III. Otherwise, those tails are determined by different configurations as is demonstrated in Section II.

The best starting point to develop the instanton formalism is the problem of white-advection passive scalar  $\theta$  since it allows for a detailed analytical treatment [12–14]. It will be shown in Section II that both  $\mathcal{P}(\theta)$  and  $\mathcal{Z}(\lambda)$  have exponential tails as it has been established before by Shraiman and Siggia [13] (see also [14]). By using this example, we shall explicitly demonstrate that different instanton configurations are responsible for the tails of the generating functional  $\mathcal{Z}(\lambda)$  at large  $\lambda$  and of PDF at large  $\theta$  respectively. It is instructive to recognize the difference between the instantons: We shall show that the instanton that is responsible for large  $\theta$  corresponds to a small strain and suppressed stretching. Contrary, the instanton that determines the tails of  $\mathcal{Z}$  corresponds to a large value of strain.

Section III presents the first step in studying instantons of the Navier-Stokes equation. Only instantons for the two-point generating functional  $\langle \exp(i\lambda(u_1 - u_2)) \rangle$  will be considered. The family of such instantons corresponds to the velocity fields with a linear spatial profile at  $r \ll L$ . Consideration of the instanton perturbations (giving the fluctuation contribution into the action) that correspond to spiral creation in the straining field of the instanton will be the subject of further publications.

## II. PASSIVE SCALAR ADVECTED BY A LARGE-SCALE VELOCITY FIELD

Let us show how the general formalism described in the Introduction works for a particular problem: the advection of a passive scalar field  $\theta(t, \mathbf{r})$  by an incompressible turbulent flow in  $d$ -dimensional space [11–14]. The advection is governed by the following equation

$$(\partial_t + v_\alpha \nabla_\alpha - \kappa \Delta) \theta = \phi, \quad \nabla_\alpha v_\alpha = 0, \quad (2.1)$$

where  $\phi(t, \mathbf{r})$  is the external source,  $\mathbf{v}$  is the advecting velocity and  $\Delta$  designates Laplacian,  $\kappa$  being the diffusion coefficient. Both  $\mathbf{v}(t, \mathbf{r})$  and  $\phi(t, \mathbf{r})$  are random functions of  $t$  and  $\mathbf{r}$ . We regard the statistics of the velocity and of the source to be independent. Therefore, all correlation functions of  $\theta$  are to be treated as averages over both statistics.

We assume that the source  $\phi$  is  $\delta$ -correlated in time and spatially correlated on a scale  $L$  and has a Gaussian statistics completely determined by the pair correlation function:

$$\langle \phi(t_1, \mathbf{r}_1) \phi(t_2, \mathbf{r}_2) \rangle = \delta(t_1 - t_2) \chi(r_{12}). \quad (2.2)$$

Here  $\chi(r_{12})$  as a function of the argument  $r_{12} \equiv |\mathbf{r}_1 - \mathbf{r}_2|$  decays on the scale  $L$ . We are interested in the behavior of the correlation functions on scales  $r \ll L$ . Thus, only the constant  $P_2 = \chi(0)$  will enter all the answers. The constant  $P_2$  has the physical meaning of the production rate of  $\theta^2$ .

Following Kraichnan [11,12], we consider the case of a Gaussian velocity  $\mathbf{v}$  delta-correlated in time and containing only large-scale space harmonics. Then the velocity statistics is also completely determined by the pair correlation function

$$\begin{aligned} \langle v_\alpha(t_1, \mathbf{r}_1) v_\beta(t_2, \mathbf{r}_2) \rangle &= \delta(t_1 - t_2) V_{\alpha\beta}, \\ V_{\alpha\beta} &= V_0 \delta_{\alpha\beta} - \mathcal{K}_{\alpha\beta}(\mathbf{r}_{12}), \quad \mathcal{K}_{\alpha\beta}(0) = 0. \end{aligned} \quad (2.3)$$

Here the so-called eddy diffusivity is as follows

$$\mathcal{K}_{\alpha\beta} = D(r^2 \delta_{\alpha\beta} - r_\alpha r_\beta) + \frac{D(d-1)}{2} \delta_{\alpha\beta} r^2, \quad (2.4)$$

where  $d$  is the dimensionality of space and isotropy of the velocity statistics being assumed. The representation (2.3,2.4) is valid for the scales less than the velocity infrared cut-off  $L_u$ , which is supposed to be the largest scale of the problem. Then  $V_0$  and  $\mathcal{K}_{\alpha\beta}$  in (2.3) are two first terms of the expansion of the velocity correlation function in  $r/L_u$  so that  $D \sim V_0/L_u^2$ . We presume also the inequality  $dDL^2 \gg \kappa$  which guarantees the existence of a convective interval of scales  $r_d \ll r \ll L$  where correlation functions of the passive scalar are formed mainly by stretching in the velocity field. Here  $r_d = 2\sqrt{\kappa/(D(d-1))}$  is the diffusion length.

The statistics of the large-scale velocity field has the remarkable property: It follows from the expressions (2.3,2.4) that the correlation function of the strain field  $\sigma_{\alpha\beta} = \nabla_\beta v_\alpha$  is  $\mathbf{r}$ -independent:

$$\langle \sigma_{\alpha\beta}(t_1) \sigma_{\mu\nu}(t_2) \rangle = D[(d+1)\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} - \delta_{\alpha\beta}\delta_{\mu\nu}] \delta(t_1 - t_2), \quad (2.5)$$

That means that the strain field  $\sigma_{\alpha\beta}$  can be treated as a random function of time  $t$  only. Just that property enables one to find in detail statistical properties of the field  $\theta$  [13,14]. To exploit the property, it is convenient to pass into the comoving reference frame that is to the frame moving with the velocity of a Lagrangian particle of the fluid. That means that we pass to the new space variable  $\mathbf{r} - \boldsymbol{\varrho}(t)$  where  $\boldsymbol{\varrho}(t)$  is the Lagrangian trajectory of the particle [5,6]. We will take the particle positioned at the origin at time  $t = 0$ , then

$$\boldsymbol{\varrho}(t) = \int_0^t d\tau \mathbf{v}(\tau, \boldsymbol{\varrho}(\tau)). \quad (2.6)$$

After the transformation  $\mathbf{r} \rightarrow \mathbf{r} - \boldsymbol{\varrho}(t)$ , the equation (2.1) acquires the form

$$\{\partial_t + [v_\alpha(t, \mathbf{r}) - v_\alpha(t, 0)] \nabla_\alpha - \kappa \Delta\} \theta = \phi, \quad (2.7)$$

It is seen from (2.3,2.4) that the statistics of  $v_\alpha(t, \mathbf{r}) - v_\alpha(t, 0)$  coincides with the statistics of  $\sigma_{\alpha\beta} r_\beta$ . That means that the generating functional corresponding to (2.7) can be written as

$$\mathcal{Z}(\lambda) = \int \mathcal{D}\theta \mathcal{D}p \mathcal{D}\sigma \exp \left( -\mathcal{F}(\sigma) + i\mathcal{I} + i \int dt d\mathbf{r} \lambda \theta \right). \quad (2.8)$$

where  $\sigma_{\alpha\beta}$  is a function of time satisfying  $\sigma_{\alpha\alpha} = 0$ , its PDF is  $\exp(-\mathcal{F})$ . The effective action  $I$  and the functional  $\mathcal{F}$  in (2.8) are

$$\begin{aligned} i\mathcal{I} = & i \int dt d\mathbf{r} (p \partial_t \theta + p \sigma_{\alpha\beta} r_\beta \nabla_\alpha \theta + \kappa \nabla p \nabla \theta) \\ & - \frac{1}{2} \int dt d\mathbf{r}_1 d\mathbf{r}_2 p_1 \chi(r_{12}) p_2, \end{aligned} \quad (2.9)$$

$$\mathcal{F} = \frac{1}{2d(d+2)D} \int dt [(d+1) \sigma_{\alpha\beta} \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \sigma_{\beta\alpha}]. \quad (2.10)$$

Note that there is the difference between (1.3) and (2.8) which is in the presence of an additional random field  $\sigma_{\alpha\beta}$ .

### A. Uniaxial Instanton

Here we examine the saddle-point contribution to the generating functional  $\mathcal{Z}(\lambda)$ . The equations describing the saddle points are extremum conditions for  $i\mathcal{I} + i \int dt d\mathbf{r} \lambda \theta - \mathcal{F}$ . Starting from the expressions (2.9,2.10) we find

$$\partial_t \theta + \sigma_{\alpha\beta} r_\beta \nabla_\alpha \theta - \kappa \nabla^2 \theta = -i \int d\mathbf{r}' \chi(|\mathbf{r} - \mathbf{r}'|) p(t, \mathbf{r}'), \quad (2.11)$$

$$\partial_t p + \sigma_{\alpha\beta} r_\beta \nabla_\alpha p + \kappa \nabla^2 p = \lambda, \quad (2.12)$$

$$\sigma_{\alpha\beta}(t) = iD \int d\mathbf{r} \left( (d+1) r_\beta \nabla_\alpha \theta - r_\alpha \nabla_\beta \theta - \delta_{\alpha\beta} r_\gamma \nabla_\gamma \theta \right) p, \quad (2.13)$$

where  $p = p(t, \mathbf{r})$ ,  $\theta = \theta(t, \mathbf{r})$ . If to take into account only the saddle-point contribution described by (2.11,2.12,2.13) then

$$\mathcal{Z}(\lambda) = \left\langle \exp \left( i \int dt d\mathbf{r} \lambda \theta \right) \right\rangle \propto \exp(-\mathcal{F}_{\text{extr}}), \quad (2.14)$$

where  $\mathcal{F}_{\text{extr}}$  is the saddle-point value of  $\mathcal{F} - i\mathcal{I} - i \int dt d\mathbf{r} \lambda \theta$ . One get from (2.9,2.10,2.12)

$$\begin{aligned} \mathcal{F}_{\text{extr}} = & \frac{1}{2} \int dt d\mathbf{r}_1 d\mathbf{r}_2 p_1 \chi(r_{12}) p_2 \\ & + \frac{1}{2d(d+2)D} \int dt [(d+1) \sigma_{\alpha\beta} \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \sigma_{\beta\alpha}]. \end{aligned} \quad (2.15)$$

In the following, we will be interested in simultaneous correlation functions of  $\theta$  so that we take the field  $\lambda$  in the form (1.8) and solve the equations for negative time  $t < 0$ . Let us stress that for the function (1.8) the term  $\lambda \theta$  is not

influenced by the transformation  $\mathbf{r} \rightarrow \mathbf{r} - \boldsymbol{\varrho}(t)$  because of  $\boldsymbol{\varrho}(0) = 0$ . Note that the system of equations (2.11,2.12,2.13) with the function (1.8) is invariant under the transformation

$$\sigma \rightarrow X\sigma, \quad p \rightarrow Xp, \quad t \rightarrow X^{-1}t, \quad y \rightarrow Xy, \quad \kappa \rightarrow X\kappa, \quad \mathcal{F}_{\text{extr}} \rightarrow X\mathcal{F}_{\text{extr}}, \quad (2.16)$$

where  $X$  is an arbitrary factor. It leads to the conclusion that

$$\mathcal{F}_{\text{extr}} = yf(y/\kappa), \quad (2.17)$$

with the function  $f$  to be determined.

We will treat nearly single-point statistics. That means that the space support of the function  $\lambda_0$  in (1.8) is taken to be much smaller than the pumping length  $L$ . From the other hand, we would like to avoid bulky formulas related to the account of diffusion. Therefore, the size of the support is believed to be much larger than the diffusion length  $r_d$ . We thus come to

$$\lambda(t, \mathbf{r}) = y\delta(t)\delta_\Lambda(\mathbf{r}), \quad (2.18)$$

where  $\delta_\Lambda(\mathbf{r})$  is a function with the characteristic size  $\Lambda^{-1}$  satisfying  $L \gg \Lambda^{-1} \gg r_d$  and normalized:  $\int d\mathbf{r} \delta_\Lambda(\mathbf{r}) = 1$ . For example, we can take

$$\delta_\Lambda(\mathbf{r}) = \frac{\Lambda^d}{\pi^{d/2}} \exp(-\Lambda^2 r^2). \quad (2.19)$$

We thus examine the following object

$$\mathcal{Z}_\Lambda = \langle \exp(iy\theta_\Lambda) \rangle, \quad (2.20)$$

where

$$\theta_\Lambda = \int d\mathbf{r} \delta_\Lambda(\mathbf{r}) \theta(t=0, \mathbf{r}). \quad (2.21)$$

Having in mind the inequality  $\Lambda^{-1} \gg r_d$ , we omit in the following the diffusive terms in the equations. The extremum conditions (2.11,2.12) are then as follows:

$$\partial_t \theta + \sigma_{\alpha\beta} r_\beta \nabla_\alpha \theta = -i \int d\mathbf{r}' \chi p', \quad (2.22)$$

$$\partial_t p + \sigma_{\alpha\beta} r_\beta \nabla_\alpha p = y\delta(t)\delta_\Lambda(\mathbf{r}), \quad (2.23)$$

It is natural to seek a solution of (2.22,2.23,2.13) in the uniaxial form what means that  $\sigma_{\alpha\beta}$  is a diagonal matrix with the components

$$\text{diag } \sigma = (-s, s/(d-1), \dots). \quad (2.24)$$

As it was suggested in [14], it is useful to pass to the new fields

$$\tilde{\theta}(t, \mathbf{r}) = \theta(t, e_\parallel x, e_\perp \mathbf{r}_\perp), \quad \tilde{p}(t, \mathbf{r}) = p(t, e_\parallel x, e_\perp \mathbf{r}_\perp), \quad (2.25)$$

where  $x$  is the coordinate along the marked direction,  $\mathbf{r}_\perp$  is the component of the radius-vector  $\mathbf{r}$  perpendicular to the direction and

$$e_\parallel(t') = \exp \left[ \int_{t'}^0 dt s(t) \right], \quad e_\perp^{d-1} = e_\parallel^{-1}. \quad (2.26)$$

Now, the equations (2.22,2.23) can be rewritten as

$$\partial_t \tilde{\theta} = -i \int d\mathbf{r}' \chi(R(t)) \tilde{p}(t, \mathbf{r}'), \quad (2.27)$$

$$\partial_t \tilde{p} = y\delta(t)\delta_\Lambda(\mathbf{r}) \rightarrow \tilde{p} = -y\delta_\Lambda(\mathbf{r}), \quad t < 0, \quad (2.28)$$

where we presented an obvious solution for  $\tilde{p}$  satisfying  $\tilde{p} = 0$  at  $t > 0$ . The quantity  $R$  in (2.27) is

$$R^2 = e_{\parallel}^2(x - x')^2 + e_{\perp}^2(\mathbf{r}_{\perp} - \mathbf{r}'_{\perp})^2. \quad (2.29)$$

Note that

$$\partial_t R = -s(x\partial_x + x'\partial'_x)R + \frac{s}{d-1}(\mathbf{r}_{\perp}\nabla_{\perp} + \mathbf{r}'_{\perp}\nabla'_{\perp})R. \quad (2.30)$$

For the considered uniaxial geometry, the relation (2.13) gives

$$s = -iD \int d\mathbf{r} \tilde{p} [(d-1)x\partial_x - \mathbf{r}_{\perp}\nabla_{\perp}] \tilde{\theta}. \quad (2.31)$$

Using now (2.27,2.28) we find

$$\partial_t s = -Dy^2 \int d\mathbf{r} d\mathbf{r}' \delta_{\Lambda}(\mathbf{r}) \delta_{\Lambda}(\mathbf{r}') [(d-1)x\partial_x - \mathbf{r}_{\perp}\nabla_{\perp}] \chi(R). \quad (2.32)$$

By virtue of (2.30) and the symmetry properties of the integrand in (2.32), we obtain

$$s\partial_t s = \frac{(d-1)Dy^2}{2} \partial_t \int d\mathbf{r} d\mathbf{r}' \delta_{\Lambda}(\mathbf{r}) \delta_{\Lambda}(\mathbf{r}') \chi(R). \quad (2.33)$$

The equation has an obvious first integral which can be established if to take into account that  $s \rightarrow 0$  if  $t \rightarrow -\infty$  [otherwise (2.15) is infinite]:

$$s^2 = (d-1)Dy^2 \int d\mathbf{r} d\mathbf{r}' \delta_{\Lambda}(\mathbf{r}) \delta_{\Lambda}(\mathbf{r}') \chi(R). \quad (2.34)$$

One can demonstrate that the main contribution to  $\mathcal{Z}_{\Lambda}$  is determined by the saddle point with  $s > 0$ . Then, in accordance with (2.26),  $e_{\parallel}$  increases with increasing  $|t|$ . That means that the characteristic value of  $R$  in (2.34) can be estimated as  $R \sim \Lambda^{-1}e_{\parallel}$ . At small  $|t|$  where  $e_{\parallel}$  is not very large,  $\chi(R)$  in (2.34) can be substituted by  $P_2 = \chi(0)$  and we find that  $s \simeq s_1$  where

$$s_1 = y\sqrt{(d-1)P_2D}. \quad (2.35)$$

That leads to  $e_{\parallel} \approx \exp(s_1|t|)$ , which is correct if  $R < L$  what means  $|t| \lesssim s_1^{-1} \ln(L\Lambda)$ . In the opposite limit  $|t| \gg s_1^{-1} \ln(L\Lambda)$ , the value of  $s$  tends to zero.

The above analysis shows that the main contribution to  $\mathcal{F}_{\text{extr}}$  (2.15) is associated with the region of integration  $|t| \lesssim s_1^{-1} \ln(L\Lambda)$  and the first term in (2.15) can be written as  $(1/2)y^2P_2s_1^{-1} \ln(L\Lambda)$ . Substituting (2.24) with (2.35) into the second term of (2.15), we find

$$\mathcal{F}_{\text{extr}} = \sqrt{\frac{P_2y^2}{(d-1)D}} \ln(L\Lambda). \quad (2.36)$$

Note that the expression is in agreement with (2.17) since we considered the case  $r_d\Lambda \ll 1$  where the answer should be  $\kappa$ -independent. It is also possible to restore  $\mathcal{Z}_{\Lambda}(y)$  in the limit  $\Lambda \rightarrow \infty$  that is for the single-point object. For this we should recognize that generally  $\mathcal{F}_{\text{extr}}$  is a function of the dimensionless parameter  $\Lambda r_d$  and use the property (2.17). Then in the limit  $r_d\Lambda \gg 1$  where  $\Lambda$ -dependence should disappear we obtain

$$\mathcal{F}_{\text{extr}} = \sqrt{\frac{P_2y^2}{(d-1)D}} \left\{ \ln(L/r_d) + \frac{1}{4} \ln \left( \frac{P_2}{D} y^2 (d-1) \right) \right\}. \quad (2.37)$$

Note the nontrivial dependence of this single-point object on  $y$ . It is the consequence of the time-dependence of the effective diffusion cut-off which could be seen at the direct solution with an explicit account of the diffusion.

Above, we considered the simplest case of the uniaxial strain matrix  $\sigma_{\alpha\beta}$ . It is not very difficult to generalize the scheme for the case where principal axes of  $\sigma_{\alpha\beta}$  are fixed (that is do not depend on time). The answer shows that it is the uniaxial solution that gives the minimum value of  $\mathcal{F}_{\text{extr}}$  and therefore only this contribution should be taken into account.

As long as we are interesting in the tail of the generating function  $\mathcal{Z}_{\Lambda}(y)$  at large  $y$ , the instanton contribution (2.36) or (2.37) gives the correct answer. However, it is not enough to consider that contribution to obtain the tails

of the PDF because the respective tail of  $\mathcal{Z}(y)$  is exponential. Indeed, we shall see below that the tails of  $\mathcal{P}(\theta)$  are determined by the contributions at moderate  $y$ . We thus face the problem of finding  $\mathcal{Z}(y)$  at arbitrary  $y$ . Fortunately, the tails of  $\mathcal{P}(\theta)$  are also determined by the instanton contribution yet this instanton is different from the above uniaxial solution, which represents the situation where stretching occurs along one marked direction. It is obvious that if the direction slowly varies in time the value of the effective action will not be essentially influenced. The role of such soft fluctuations is expected to be negligible if the characteristic time  $s_1^{-1} \ln(L\Lambda)$  of the stretching is small enough. We thus conclude, taking into account (2.35), that the expression (2.36) is correct at large  $y$ . At moderate  $y$ , the fluctuations of the stretching direction should be taken into account, it is the topic of the next subsection. There, we shall explicitly integrate over soft mode and obtain different equations for the instanton.

## B. Isotropic Instanton

Here, we are going to take into account the fluctuations of the stretching direction which were neglected in the preceding subsection. For that purpose, it is useful to introduce the variable measuring the stretching rate along the current stretching direction (the direction of the maximal Lyapunov exponent) determined by the strain field  $\sigma_{\alpha\beta} = \nabla_\beta v_\alpha$ . For this aim, it is useful to perform the transformation of the fields  $\theta, p$  generalizing (2.25) for an arbitrary  $\sigma_{\alpha\beta}$  [14]. Namely, let us pass to the fields

$$\tilde{\theta}(t, \mathbf{r}) = \theta(t, M_{\alpha\beta} r_\beta), \quad \tilde{p}(t, \mathbf{r}) = p(t, M_{\alpha\beta} r_\beta), \quad (2.38)$$

with  $d \times d$  matrix  $M_{\alpha\beta}$  controlled by the equation

$$\partial_t \hat{M} = \hat{\sigma} \hat{M}, \quad \hat{M}(t=0) = \hat{1}, \quad (2.39)$$

with a formal solution

$$\hat{M} = \text{T exp} \left( \int_0^t dt' \hat{\sigma}(t') \right). \quad (2.40)$$

The symbol T designates the anti-chronological ordering for negative  $t$ . Note that  $\det \hat{M} = 1$  due to the incompressibility condition  $\text{tr} \hat{\sigma} = \nabla_\alpha v_\alpha = 0$ . Performing the substitution in (2.9) and passing to the new space variable  $\hat{M}\mathbf{r}$  (the Jacobian of the transformation is equal to unity due to  $\det \hat{M} = 1$ ), one obtains

$$i\mathcal{I} = i \int dt d\mathbf{r} \tilde{p} \partial_t \tilde{\theta} - \frac{1}{2} \int dt d\mathbf{r}_1 d\mathbf{r}_2 \tilde{p}_1 \chi(R) \tilde{p}_2, \quad (2.41)$$

where

$$R_\alpha = M_{\alpha\beta} (r_{1\beta} - r_{2\beta}). \quad (2.42)$$

We see that only  $\mathbf{R}$  is  $\sigma$ -dependent in (2.41) and, moreover, only its absolute value  $R$  enters the effective action. Just that value is a measure of the stretching irrespective of the directions of the current main axes of the matrix  $\hat{\sigma}$ . The statistics of  $R$  can be established starting from the PDF  $\exp(-\mathcal{F})$ , see e.g. [15]. The answer is that, for negative times,  $R$  can be written as

$$R(t) = \exp \left( \int_t^0 dt' \zeta(t') \right) |\mathbf{r}_1 - \mathbf{r}_2|, \quad (2.43)$$

with the random variable  $\zeta$  having PDF  $\exp(-\mathcal{F}_\zeta)$  with

$$\mathcal{F}_\zeta = \int dt \frac{1}{2D(d-1)} \left( \zeta - \frac{d(d-1)}{2} D \right)^2. \quad (2.44)$$

The generating functional (2.20) is thus rewritten as

$$\mathcal{Z}_\Lambda = \int \mathcal{D}\tilde{\theta} \mathcal{D}\tilde{p} \mathcal{D}\zeta \exp(iy\theta_\Lambda + i\mathcal{I} - \mathcal{F}_\zeta), \quad (2.45)$$

where  $\theta_\Lambda$  is defined by (2.21).

We have performed the exact transformation of the statistical weight. Let us stress that the situation described by (2.45) is isotropic from the beginning whereas the solution found in the preceding subsection was anisotropic. Now, we formulate extremum conditions describing a saddle point for the argument of the exponent in (2.45):

$$\partial_t \tilde{p} = y \delta(t) \delta_\Lambda(\mathbf{r}), \quad (2.46)$$

$$\partial_t \tilde{\theta}(t, \mathbf{r}_1) = -i \int d\mathbf{r}_2 \chi[R(t)] \tilde{p}(t, \mathbf{r}_2), \quad (2.47)$$

$$\zeta(t') = \frac{d(d-1)}{2} D - \frac{d-1}{2} D \int_{-\infty}^{t'} dt d\mathbf{r}_1 d\mathbf{r}_2 \tilde{p}_1 \tilde{p}_2 \frac{\partial \chi}{\partial R} R. \quad (2.48)$$

The equation (2.46) has the same form as (2.28) and has consequently the same solution  $\tilde{p} = -y \delta_\Lambda(\mathbf{r})$ . It follows from (2.45, 2.46) that in the saddle-point approximation  $\mathcal{Z}_\Lambda \propto \exp(-\mathcal{F}_{\text{extr}})$  where

$$\mathcal{F}_{\text{extr}} = \frac{1}{2} \int dt d\mathbf{r}_1 d\mathbf{r}_2 \tilde{p}_1 \tilde{p}_2 \chi(R) + \frac{1}{2D(d-1)} \int dt \left( \zeta - \frac{(d-1)d}{2} D \right)^2. \quad (2.49)$$

It follows from (2.43) that  $\partial_t R = -\zeta R$ . Using that, we can find the first integral of the equation (2.48):

$$\zeta^2 = \frac{d^2(d-1)^2}{4} D^2 + (d-1) D y^2 \int d\mathbf{r}_1 d\mathbf{r}_2 \delta_\Lambda(\mathbf{r}_1) \delta_\Lambda(\mathbf{r}_2) \chi(R). \quad (2.50)$$

The constant here is established using the property  $\zeta \rightarrow (d-1)dD/2$  at  $t \rightarrow -\infty$  following from  $\theta \rightarrow 0$  at  $t \rightarrow -\infty$  [(2.49) is infinite otherwise]. The characteristic value of  $R$  in the right-hand side of (2.50) can be estimated as

$$R(t') \sim \Lambda^{-1} \exp \int_{t'}^0 dt \zeta(t). \quad (2.51)$$

If  $R \ll L$  then the integral in the right-hand side of (2.50) is approximately equal to  $P_2$ , if  $R \gg L$  then the integral is negligible. That means that there are two different time intervals. At large  $|t|$ , it is  $\zeta \simeq (d-1)dD/2$  and at small  $|t|$  it is  $\zeta \simeq \zeta_1$  where

$$\zeta_1^2 = \frac{(d-1)^2 d^2}{4} D^2 + (d-1) D y^2 P_2. \quad (2.52)$$

The boundary between the regions is at  $|t| \sim \zeta_1^{-1} \ln(L\Lambda)$ . The main contribution to  $\mathcal{F}_{\text{extr}}$  (2.49) is associated with the region  $|t| < t_1 = \zeta_1^{-1} \ln(L\Lambda)$ . The first term in (2.49) can be substituted by  $y^2 P_2 t_1 / 2$  and the second one can be substituted by  $(2D(d-1))^{-1} (\zeta_1 - (d-1)dD/2)^2 t_1$ . Using (2.52) we find

$$\mathcal{F}_{\text{extr}} = \left( \sqrt{\frac{d^2}{4} + \frac{P_2 y^2}{(d-1)D}} - \frac{d}{2} \right) \ln(L\Lambda). \quad (2.53)$$

Comparing the expression (2.53) with (2.36), we conclude that fluctuations of the stretching direction can be neglected if  $y^2 \gg D d^3 P_2^{-1}$ . Let us stress that at  $y^2 \sim D d^3 P_2^{-1}$  the value of (2.53) is much larger than unity. That means that violation of (2.36) is not associated with destructing saddle-point regime, it is rather related to an incorrect calculation of soft fluctuations in the saddle-point regime. Note also that the role of fluctuations increases with increasing the space dimensionality  $d$ .

### C. Probability Distribution Function

The scheme proposed in the preceding subsections can be applied also to calculating PDF  $\mathcal{P}_\Lambda(\vartheta)$  of the quantity  $\theta_\Lambda$  (2.21). Let us start from the average

$$\langle \theta_\Lambda^{2n} \rangle = \int \mathcal{D}\theta \mathcal{D}p \mathcal{D}\sigma \exp(i\mathcal{I} - \mathcal{F}_\sigma + 2n \ln \theta_\Lambda). \quad (2.54)$$



Thus we see that the saddle-point contribution to  $\langle \theta_\Lambda^{2n} \rangle$  is determined by extrema of  $i\mathcal{I} - \mathcal{F}_\sigma + 2n \ln \theta_\Lambda$  which coincide with (2.11,2.12,2.13) if to substitute

$$y \rightarrow -\frac{2ni}{\theta_\Lambda}. \quad (2.55)$$

Then an attempt to find the analog of the uniaxial instanton fails. The formal reason for this is in additional  $i$  in (2.55). The physical reason is that the uniaxial instanton is an adequate object for the statistics of fast processes whereas  $\langle \theta_\Lambda^{2n} \rangle$  is determined by slow processes.

To find a solution, we should pass to the isotropic instanton. That means that we should perform the same transformation of the fields as in the preceding subsection what leads to the saddle-point equations (2.46,2.47,2.48) with (2.55). The equations has a solution of the same type as considered above with

$$\zeta_1^2 = \frac{d^2(d-1)^2}{4} D^2 - (d-1)DP_2 \frac{4n^2}{\theta_\Lambda^2}. \quad (2.56)$$

The value of  $\theta_\Lambda$  in (2.56) is the parameter which can be found from the equation analogous to (2.47) which now reads

$$\partial_t \tilde{\theta} = \frac{2n}{\theta_\Lambda} \int d\mathbf{r}_2 \chi(R) \delta_\Lambda(\mathbf{r}_2). \quad (2.57)$$

As previously, the integral in the right-hand side of (2.57) for  $\mathbf{r}_1 = 0$  is equal to  $P_2$  if  $|t| \lesssim \ln(L\Lambda)/\zeta_1$  and is negligible otherwise. We thus come to the conclusion that

$$\theta_\Lambda^2 \simeq \theta^2(t=0, \mathbf{r}=0) \simeq 2nP_2 \frac{\ln(L\Lambda)}{\zeta_1}. \quad (2.58)$$

Substituting the relation into (2.56) we find the equation on  $\zeta_1$  leading to

$$\zeta_1 = (d-1)D \left\{ -\frac{n}{\ln(L\Lambda)} + \sqrt{\frac{n^2}{\ln^2(L\Lambda)} + \frac{d^2}{4}} \right\}. \quad (2.59)$$

We see that  $\zeta_1$  decreases with increasing  $n$  and consequently the characteristic time  $\ln(L\Lambda)/\zeta_1$  increases with increasing  $n$ . Substituting now (2.59) into (2.58) one obtains

$$\theta_{\Lambda n}^2 = \frac{8nP_2 \ln(L\Lambda)}{d^2 D(d-1)} \left\{ \frac{n}{\ln(L\Lambda)} + \sqrt{\frac{n^2}{\ln^2(L\Lambda)} + \frac{d^2}{4}} \right\}. \quad (2.60)$$

It is not very difficult to recognize that the main contribution to the saddle-point value of  $i\mathcal{I} - \mathcal{F}_\zeta + 2n \ln \theta_\Lambda$  is determined by the last term. That means that

$$\langle \theta_\Lambda^{2n} \rangle \propto \exp(-\mathcal{F}_{\text{extr}}) \propto \theta_{\Lambda n}^{2n}, \quad (2.61)$$

with  $\theta_{\Lambda n}$  from (2.60).

The same result can be deduced by the alternative method. Namely, starting from (2.53) we can calculate the tail of the PDF  $\mathcal{P}_\Lambda(\vartheta)$  for the quantity  $\theta_\Lambda$  (2.21). The function  $\mathcal{P}_\Lambda(\vartheta)$  is the Fourier transform of  $\mathcal{Z}_\Lambda(y)$ :

$$\mathcal{P}_\Lambda(\vartheta) = \int dy \exp(-iy\vartheta) \mathcal{Z}_\Lambda(y) \propto \int dy \exp(-iy\vartheta - \mathcal{F}_{\text{extr}}). \quad (2.62)$$

Substituting here (2.49) and calculating the integral over  $y$  by the saddle-point method [15] we find

$$\mathcal{P}_\Lambda(\vartheta) \propto \exp \left\{ \frac{d}{2} \ln(L\Lambda) \left( 1 - \sqrt{1 + \frac{d-1}{P_2} D \frac{\vartheta^2}{\ln^2(L\Lambda)}} \right) \right\}, \quad (2.63)$$

what is in agreement with [13,15,14]. Formally, the expression (2.63) is valid at  $\vartheta \rightarrow \infty$  but really it covers the whole region of  $\vartheta$  because the PDF is Gaussian at small  $\vartheta$  [14]. The distant tails of the PDF are exponential as has been established first by Shraiman and Siggia [13]. Note that the value of the Lyapunov exponent  $\zeta_1$  corresponding to the saddle point in (2.62) is

$$\zeta_{\text{extr}} = \frac{d}{2}(d-1)D \left( 1 + \frac{d-1}{P_2} D \frac{\vartheta^2}{\ln^2(L\Lambda)} \right)^{-1/2}. \quad (2.64)$$

We see that the value decreases with increasing  $\vartheta$  whereas the value  $\zeta_1$  (2.52) increases with increasing  $y$ . Note also that the value of  $y$  corresponding to the extremum point is

$$y_{\text{extr}}^2 = -\frac{(d-1)d^2D}{4P_2} \frac{\vartheta^2}{\vartheta^2 + P_2/((d-1)D)}. \quad (2.65)$$

That means that  $|y_{\text{extr}}^2| < d^3D/P_2$  and consequently the extremum point lies beyond the applicability region of the approximation (2.36). This is the reason why (2.36) does not admit to restore  $\mathcal{P}_\Lambda(\vartheta)$ .

Now, we can calculate  $\langle \theta_\Lambda^{2n} \rangle$  starting from the definition

$$\langle \theta_\Lambda^{2n} \rangle = \int d\vartheta \vartheta^{2n} \mathcal{P}_\Lambda(\vartheta). \quad (2.66)$$

This integral can be calculated using again the saddle-point method. The result coincides, of course, with (2.61). We thus conclude that (2.49) or (2.63) cover both cases of slow and of fast processes. That means that the account of fluctuations in the direction of stretching (performed in the preceding subsection) give us the tool for finding tails of both PDF and of the generating functional.

#### D. Discussion

We considered the statistics of the passive scalar advected by the random velocity field in the framework of the instanton formalism. The consideration was very instructive since it revealed some nontrivial peculiarities of the formalism. First of all, we see that the direct solution of the saddle-point equations gives us the answer which satisfactory describes the tail of the generating functional  $\mathcal{Z}(\lambda)$  but cannot serve to restore the tail of PDF  $\mathcal{P}(\vartheta)$ . The physical reason for this lies in difference of processes forming the tails: The tail by  $\mathcal{Z}$  is related to the fast processes with the characteristic time decreasing as  $\lambda$  increases while the tail by  $\mathcal{P}$  is related to slow processes with characteristic time increasing as  $\vartheta$  increases. The conclusion can be directly extracted from (2.35) and (2.64). For the slow processes, the fluctuations of the stretching direction are relevant which do not destroy the saddle-point (instanton) regime but renormalize the naive answer. For that particular problem, the fluctuations can be explicitly taken into account after the special transformations of the fields. Although the trick cannot be widely generalized it shows the direction of improving naive answers. In the general case, we expect that the direct solution of the saddle-point equations will produce nonsymmetric instantons with a degeneracy parameter (like the direction of the marked axis in the considered case). Then, there exists the “Goldstone” mode related to slightly nonhomogeneous variations of the parameter. Such mode is soft since it only weakly disturbs the naive instantons. Therefore, the fluctuations related to the soft mode are relevant and should be explicitly taken into account.

### III. THE SIMPLEST INSTANTON OF AN INCOMPRESSIBLE VELOCITY FIELD

Here, we describe the first step in considering a much more complicated problem of finding the tails of the PDF for velocity field in three-dimensional incompressible turbulence. We consider two-point statistics and show that an instanton with a linear spatial profile naturally appears as a basic flow.

The effective action (1.4) for the Navier-Stokes equation can be written as follows

$$\begin{aligned} \mathcal{I} = & \int dt d\mathbf{r} (p_\alpha \partial_t v_\alpha + p_\alpha v_\beta \nabla_\beta v_\alpha - \nu p_\alpha \nabla^2 v_\alpha + p_\alpha \nabla_\alpha P + Q \nabla_\alpha v_\alpha) \\ & + \frac{i}{2} \int dt dt' d\mathbf{r} d\mathbf{r}' \Xi(t-t', \mathbf{r}-\mathbf{r}') p_\alpha p'_\alpha, \end{aligned} \quad (3.1)$$

The additional independent fields  $P$  and  $Q$  play the role of Lagrange multipliers enforcing the incompressibility conditions  $\nabla_\alpha v_\alpha = 0$  and the analogous condition  $\nabla_\alpha p_\alpha = 0$  for the response field  $p_\alpha$ . The field  $P$  has the meaning of pressure (divided by the mass density  $\rho$ ). The origin of the terms with the fields  $P, Q$  in the effective action is related to the continuity equation  $\partial_t \rho + \nabla_\alpha (\rho v_\alpha) = 0$ , which should be incorporated into the effective action,  $Q$  is just the auxiliary (response) field corresponding to the equation. At the condition that all velocities are much smaller

than the sound velocity, it is possible to neglect the time derivative in  $\partial_t \rho + \nabla_\alpha (\rho v_\alpha) = 0$  and variations of the mass density what leads to the term  $Q \nabla_\alpha v_\alpha$  in (3.1). While variations of the mass density can be neglected variations in the pressure are relevant. Therefore, it is natural to pass from the integration over the mass density to the integration over the pressure as it is implied in (3.1).

We are going to examine the generating functional for the velocity

$$\begin{aligned} \mathcal{Z}(\boldsymbol{\lambda}) &\equiv \left\langle \exp \left( i \int dt d\mathbf{r} \boldsymbol{\lambda} \mathbf{v} \right) \right\rangle \\ &= \int \mathcal{D}p \mathcal{D}v \mathcal{D}P \mathcal{D}Q \exp \left( i\mathcal{I} + i \int dt d\mathbf{r} \boldsymbol{\lambda} \mathbf{v} \right). \end{aligned} \quad (3.2)$$

The extremum conditions for the argument of the exponent in (3.2) determining the Navier-Stokes instanton read

$$\partial_t v_\alpha(\mathbf{r}) + v_\beta(t, \mathbf{r}) \nabla_\beta v_\alpha(t, \mathbf{r}) - \nu \nabla^2 v_\alpha(t, \mathbf{r}) + \nabla_\alpha P(t, \mathbf{r}) = -i \int dt' \int \frac{d^d k}{(2\pi)^d} \exp(i\mathbf{k}\mathbf{r}) \Xi(t - t', \mathbf{k}) p_\alpha(t', \mathbf{k}), \quad (3.3)$$

$$\partial_t p_\alpha(t, \mathbf{r}) - p_\beta(t, \mathbf{r}) \nabla_\alpha v_\beta(t, \mathbf{r}) + v_\beta(t, \mathbf{r}) \nabla_\beta p_\alpha(t, \mathbf{r}) + \nu \nabla^2 p_\alpha(t, \mathbf{r}) + \nabla_\alpha Q(t, \mathbf{r}) = \lambda_\alpha(t, \mathbf{r}), \quad (3.4)$$

where  $\Xi(\mathbf{k})$  and  $p_\alpha(\mathbf{k})$  are Fourier transforms of  $\Xi(\mathbf{r})$  and  $p_\alpha(\mathbf{r})$  respectively. In (3.3,3.4) the conditions  $\nabla_\alpha v_\alpha = 0$ ,  $\nabla_\alpha p_\alpha = 0$  are also implied which originate from varying over the fields  $P$  and  $Q$ . Then the values of the fields  $P$  and  $Q$  can also be found from the conditions. It gives the relations

$$\nabla^2 P = -\nabla_\alpha (v_\beta \nabla_\beta v_\alpha). \quad (3.5)$$

$$\nabla^2 Q = \nabla_\alpha (p_\beta \nabla_\alpha v_\beta - v_\beta \nabla_\beta p_\alpha). \quad (3.6)$$

Note that similar idea of the instanton formalism has been discussed also by Giles [17] yet his approach is based on uncontrollable approximations.

In the following we consider the simultaneous correlation functions of the velocity differences  $\langle [\mathbf{v}(0, \boldsymbol{\rho}/2) - \mathbf{v}(0, -\boldsymbol{\rho}/2)]^{2n} \rangle$  where  $\boldsymbol{\rho}$  is the separation between the points. The functional generating such functions is extracted from  $\mathcal{Z}(\boldsymbol{\lambda})$  if one gets

$$\lambda_\alpha = y n_\alpha \delta(t) [\delta(\mathbf{r} - \boldsymbol{\rho}/2) - \delta(\mathbf{r} + \boldsymbol{\rho}/2)], \quad (3.7)$$

where  $\mathbf{n}$  is a unit vector. As was explained in Introduction the presence of such term in right-hand side of (3.4) means that we should solve the problem at negative times  $t$  with the final condition

$$p_\alpha(0, \mathbf{r}) = -y(\delta_{\alpha\beta} - \nabla_\alpha \nabla_\beta \nabla^{-2}) n_\beta [\delta(\mathbf{r} - \boldsymbol{\rho}/2) - \delta(\mathbf{r} + \boldsymbol{\rho}/2)]. \quad (3.8)$$

We assume that the pumping correlation function  $\Xi$  is delta-correlated in time:  $\Xi(t, \mathbf{r}) = \delta(t) \chi(\mathbf{r})$ . Then the system of equations (3.3-3.6) is invariant under the transformation analogous to (2.17)

$$t \rightarrow X^{-1}t, \quad \mathbf{v} \rightarrow X\mathbf{v}, \quad P \rightarrow X^2P, \quad Q \rightarrow X^3Q, \quad \nu \rightarrow X\nu, \quad \lambda \rightarrow X\lambda, \quad \mathbf{p} \rightarrow X^3\mathbf{p}, \quad (3.9)$$

where  $X$  is an arbitrary factor. For the function (3.8) the transformation (3.9) means  $y \rightarrow X^2y$ . The extremum value  $\mathcal{F}_{\text{extr}}$  of the argument of the exponent in (3.2) transforms as  $\mathcal{F}_{\text{extr}} \rightarrow X^3\mathcal{F}_{\text{extr}}$  at (3.9). That leads to the conclusion that

$$\mathcal{Z}(y) \propto \exp(-\mathcal{F}_{\text{extr}}) \quad \mathcal{F}_{\text{extr}} = y^{3/2} f(y/\nu^2), \quad (3.10)$$

with the function  $f$  to be determined. We expect that in the limit  $y \rightarrow \infty$  a  $\nu$ -dependence in the function disappears. Then we conclude  $\mathcal{F}_{\text{extr}} \propto y^{3/2}$ .

The characteristic wave vector  $k_0$  in the correlation function  $\chi(\mathbf{k})$  of the pumping force is of the order of the inverse pumping length  $L$ . Then examining the region  $\mathbf{r} \ll L$  one can expand the exponent  $\exp(i\mathbf{k}\mathbf{r})$  in (3.3) into the series over  $\mathbf{k}\mathbf{r}$ . The first term of the expansion produces the zero contribution to the right-hand side of (3.3) because of the structure of the field  $p$  determined by the condition (3.8): The condition means that at  $t = 0$   $\mathbf{p}(\mathbf{r}) = -\mathbf{p}(-\mathbf{r})$ , the property is reproduced by the equations, so that  $\mathbf{p}(\mathbf{k}) = -\mathbf{p}(-\mathbf{k})$  at any time  $t$ . Thus the leading term of the expansion of the right-hand side is linear in  $\mathbf{r}$ . That means that the equation (3.3) admits as a solution in the region  $|\mathbf{r}| \ll L$  a linear profile

$$v_\alpha = \sigma_{\alpha\beta}(t) r_\beta, \quad \sigma_{\alpha\alpha} = 0. \quad (3.11)$$

Then we obtain from (3.3)

$$\partial_t \sigma_{\alpha\gamma} + \sigma_{\alpha\beta} \sigma_{\beta\gamma} - \frac{1}{d} \delta_{\alpha\gamma} (\sigma_{\mu\nu} \sigma_{\nu\mu}) = \int \frac{d^d k}{(2\pi)^d} k_\gamma p_\alpha(\mathbf{k}) \chi(\mathbf{k}). \quad (3.12)$$

Here we substituted the expression for the pressure

$$P = -\frac{1}{d} (\sigma_{\mu\nu} \sigma_{\nu\mu}) r^2, \quad (3.13)$$

which provides for the condition  $\nabla_\alpha v_\alpha = 0$ . Note that  $P$  is defined up to a harmonic function, the expression (3.13) is chosen because of its isotropy.

For the linear velocity profile the equation (3.4) can be rewritten in Fourier representation as

$$\partial_t p_\alpha - \sigma_{\beta\alpha} p_\beta - \sigma_{\beta\gamma} k_\beta \frac{\partial}{\partial k_\gamma} p_\alpha - \nu k^2 p_\alpha + i k_\alpha Q = 0 \quad (3.14)$$

which should be solved with the condition following from (3.8):

$$p_\alpha(t=0, \mathbf{k}) = 2iy \left( \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) n_\beta \sin(\mathbf{k}\boldsymbol{\rho}/2). \quad (3.15)$$

The characteristic wave vector  $\mathbf{k}$  in (3.12) is of the order of  $L^{-1}$ . Thus we can expand  $\sin(\mathbf{k}\boldsymbol{\rho}/2)$  in  $\mathbf{k}\boldsymbol{\rho}$  and keep only the first nonvanishing term of the expansion  $\propto \mathbf{k}\boldsymbol{\rho}$ . As was discussed in the Introduction, the response field  $\mathbf{p}(\mathbf{r}, t)$  propagates backwards in time, starting with the initial value (3.15) at  $t = 0$ . We shall see that for a long time (determined by a small viscosity) the field  $\mathbf{p}(t, \mathbf{k})$  at  $k \sim L^{-1}$  has the same structure  $\propto \mathbf{k}\boldsymbol{\rho}$ .

There is a general family of the flows with linear profiles – see Sect.III C below. We start by considering the simplest case. We assume below that the point separation  $\boldsymbol{\rho}$  is directed along the same vector  $n_\alpha$  as the measured velocity components:  $\rho_\alpha = n_\alpha \rho$ . Then the problem possesses the axial symmetry which allows us to look for the following uniaxial strain matrix

$$\sigma_{\alpha\beta} = s (\delta_{\alpha\beta} - d n_\alpha n_\beta). \quad (3.16)$$

The same symmetry admits the ansatz

$$p_\alpha(t, \mathbf{k}) = \left( \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) i y n_\beta \phi(t, z) \mathbf{k} n_\rho, \quad (3.17)$$

correct in the limit of small  $k$ . In (3.17)  $z = \mathbf{k}\mathbf{n}/k$  and the function  $\phi(t, z)$  to be found has the initial (final) value  $\phi(t=0, z) = 1$ . Substituting (3.15,3.17) into (3.14) we find  $Q = 0$  and

$$s^{-1} (\partial_t - \nu k^2) \phi + dz (1 - z^2) \partial_z \phi + 2[(d-1) - dz^2] \phi = 0. \quad (3.18)$$

Substituting the expression (3.17) into the right-hand side of (3.12) we find

$$\int \frac{d^d k}{(2\pi)^d} k_\gamma p_\alpha(\mathbf{k}) \chi(\mathbf{k}) = G (d n_\alpha n_\gamma - \delta_{\alpha\gamma}), \quad (3.19)$$

where

$$G = i y \rho C \int_{-1}^1 dz z^2 (1 - z^2)^{(d-1)/2} \phi(t, z), \quad (3.20)$$

and

$$C = \frac{S_{d-1}}{(2\pi)^d (d-1)} \int_0^\infty dk k^{d+1} \chi(k). \quad (3.21)$$

Here  $S_d$  is the area of the unit sphere in  $d$ -dimensional space  $S_d = 2\pi^{d/2}/\Gamma(d/2)$ . The constant  $C$  can be estimated as  $C \sim \mathcal{E}/L^2$  where  $\mathcal{E} = \langle v_\alpha \partial_t v_\alpha \rangle$  is the energy dissipation rate. Substituting now (3.16,3.19) into (3.12) we find

$$\partial_t s = (d-2)s^2 - G. \quad (3.22)$$

Our next problem is to find  $G$  as a functional of  $s$ , to close this set of equations. We have to solve the equation (3.18) for  $\phi$ . Here, viscous and inviscid cases are slightly different. We start from considering an inviscid Euler equation, then we will account for the viscosity.

### A. Instanton of the Euler equation

Neglecting viscosity in (3.18) we get a general solution

$$\phi = h^2 z^{-2} F\left(\frac{z^2 h^{-2d}}{1 - z^2}\right), \quad h(t) = \exp\left(\int_0^t s(t') dt'\right). \quad (3.23)$$

The initial condition  $\phi(0, z) = 1$  fixes the function  $F$ :

$$\phi(z) = \frac{h^{2-2d}}{1 - z^2 + z^2 h^{-2d}}. \quad (3.24)$$

We get the system of equations

$$\dot{s} = (d-2)s^2 - G(h), \quad \dot{h} = sh. \quad (3.25)$$

This system for the variable  $q = h^{2-d}$  becomes the usual potential problem  $\ddot{q} = -U'(q)$  with the potential

$$U(q) = -(d-2) \int dq q G\left(q^{\frac{1}{2-d}}\right).$$

For  $d = 3$ ,

$$G(h) = \frac{iy\rho C h^2}{h^{2d} - 1} \left[ \frac{2}{1 - h^{-2d}} - \frac{1}{3} - \frac{\ln(2h^{2d} - 1)}{h^{2d} - 1} \right]$$

The relevant solution, which vanishes at  $t = -\infty$ , corresponds to the zero energy in this potential [ $s(t) \propto -1/t$  as  $t \rightarrow -\infty$ ]. Therefore,  $\dot{q}^2 = 2U(0) - 2U(q)$  and

$$\dot{q}_{t=0} = \sqrt{2[U(0) - U(1)]} = C_1 \sqrt{\mathcal{E} y \rho} / L. \quad (3.26)$$

Then, the strain at the moment  $t = 0$  becomes  $\sigma_{\alpha\beta} = \dot{q}(\delta_{\alpha\beta} - dn_{\alpha}n_{\beta})/q(2-d)$ . In accordance with (1.7) the logarithmic derivative of  $\mathcal{Z}$ -functional is related to the average initial value of the velocity difference  $\mathcal{Z}'(y)/\mathcal{Z}(y) = n_{\alpha}\langle v_{\alpha}(\rho, 0) - v_{\alpha}(-\rho, 0) \rangle$ . In the leading WKB approximation at large  $y$  this average can be replaced by the contribution from the instanton solution:

$$(\ln \mathcal{Z})'(y) = 2\rho n_{\alpha}n_{\beta}\sigma_{\alpha\beta} = \frac{2\dot{q}(d-1)}{q(d-2)} = C_2 \sqrt{\mathcal{E} y \rho^3 L^{-2}}. \quad (3.27)$$

Finally, we obtain the surprisingly simple result

$$\mathcal{Z}(y) \propto \exp\left[C_3 \sqrt{\mathcal{E}(y\rho)^3 L^{-2}}\right]. \quad (3.28)$$

with the dimensionless constants  $C_1, C_2$  and  $C_3$  to be calculated. This result is in agreement with the general form (3.10), it contains also  $\rho$ -dependence.

### B. Account of viscosity

When the viscous terms are kept, the solution is modified as follows. With the same assumption  $k_0\rho \simeq \rho/L \ll 1$ , we can still look for the uniform strain solution. The viscosity will drop from the velocity equation, but not from the response field equation (3.14). The extra term  $\nu k^2 p$  can be compensated by extra time dependent exponential

$$p_{\alpha}(t, \mathbf{k}) = \left(\delta_{\alpha\beta} - \frac{k_{\alpha}k_{\beta}}{k^2}\right) i y n_{\beta} \phi(t, z) \mathbf{k} n_{\rho} \exp[\nu R(t, z) k^2]. \quad (3.29)$$

The balance of  $k^0$  and  $k^1$  terms in the equation is the same as before. The balance of  $k^2$  terms gives the equation

$$\dot{R} = 1 + s\hat{L}R = 1 - dsz(1 - z^2)\frac{\partial R}{\partial z} + 2s(1 - dz^2)R \quad (3.30)$$

with the boundary condition  $R(0) = 0$ . The substitution  $R(z, t) = A(t) + B(t)z^2$  reduces the PDE to two ODEs

$$\begin{aligned}\dot{A} &= 1 + 2As \\ \dot{B} &= 2s(B - Bd - Ad) .\end{aligned}$$

The solution is expressed via  $s(t)$ ; at  $t \rightarrow -\infty$  it grows linearly:  $R \approx t$ . The influence of the viscosity on our solution is weak, it smears the peaks at  $p$  and manifests itself when  $\nu R \simeq L^2$  i.e. at  $t \simeq L^2/\nu$ . That time should be much larger than the time of instanton formation  $\sqrt{L^2/\mathcal{E}y\rho}$ . Our asymptotic expression (3.28) is insensitive to viscosity if  $\nu \ll L\sqrt{\mathcal{E}y\rho}$ .

### C. Instanton family

Considering more general strain does not change basic conclusions of this section. Let us describe, for instance, a general three-dimensional symmetric flow of the type considered in [18]. In the cylindrical coordinates with  $z$ -axis along  $\boldsymbol{\rho}$ , the velocity vector field at  $r \ll L$  is given by

$$\mathbf{u} = (u_r, u_\theta, u_z) = (-\sigma r/2, \omega r/2, \sigma z) . \quad (3.31)$$

Here vorticity has only  $z$ -component  $\omega(t)$  which is a function of time as well as strain  $\sigma(t)$ . The pressure is now of the form

$$P = -gr^2 - e[r^2 \sin \theta \cos \theta + rz(\sin \theta + \cos \theta)] .$$

Particular details of the solution depend on the relation between  $g$  and  $e$ . The diagonal elements (proportional to  $g$ ) are determined locally from Poisson equation  $\Delta P = -\text{div}(\mathbf{u} \cdot \nabla \mathbf{u})$ . Note that the off-diagonal pressure elements are generally determined by the global structure of the flow. In our case, the value of  $e$  is determined by the distant asymptotics  $u \rightarrow 0, P \rightarrow \text{const}$  at  $r \rightarrow \infty$  and matching conditions at  $r \simeq L$ , which depends on the particular choice of the pumping  $\chi$ . The global description of the flows for the whole instanton family is still ahead of us. As far as the functional dependence of the respective  $\mathcal{Z}(y, \boldsymbol{\rho})$  is concerned, it is the same for the whole family and does not depend of the large-scale behavior of the pumping. Considering, for instance, the case  $g = 0$  [opposite to the diagonal case (3.13) considered before], we get  $\omega = \sqrt{3}s$  and the system of equations similar to (3.25)

$$\dot{s} = s^2 - G'(h), \quad \dot{h} = sh$$

with another yet qualitatively similar function  $G'$ . For the variable  $q(t)$ , related to  $s(t)$  by  $\dot{q} = -sq$ , the Newton equation appears with a potential energy  $U$  that allow for a single solution (zero energy separatrix) vanishing as  $s(t) \propto 1/t$  at  $t \rightarrow -\infty$ . The basic result  $\ln \mathcal{Z}(y, \rho) \propto (y\rho)^{3/2} \sqrt{\mathcal{E}}/L$  is valid for the whole family in agreement with (3.10).

### D. Discussion

The particular instanton found has the scaling

$$\delta u(\rho) = u(\rho/2) - u(-\rho/2) \propto \rho , \quad (3.32)$$

it would give the asymptotics of the right tail of the PDF  $\mathcal{P}(\delta u, \rho) \propto \exp[-(\delta u/\rho)^3]$ . It is unclear at the moment if there are flows where such an asymptotics takes place; most probably, this simplest instanton does not realize the main extremum of the action. Note that the similar instanton with the linear profile is found for the Burgers problem [16] where it indeed determines the right tail ( $\delta u > 0$ ) of velocity PDF due to sawtooth waves. The general analysis of the whole family of instanton solutions for the two-point velocity statistics at the framework of the Navier-Stokes equation will be published elsewhere. Also, the crucial problem of the contribution to the action from the fluctuations against the instanton background will be considered. It is clear that, in the straining flow of the instanton, any vorticity perturbation produces a spiral with the accumulation point at the velocity null. The scaling of the perturbation contribution is different from (3.32); for instance, it will give Kolmogorov's 5/3-law for the pair correlation function as in the Lundgren example [19]. The analysis of the instanton fluctuations will be the subject of further publications. Note that the instanton formalism provides a natural (and long-expected) tool for incorporating numerous results on particular solutions of the Navier-Stokes equations into the statistical theory of turbulence.

## ACKNOWLEDGMENTS

We are grateful to E. Balkovsky and M. Chertkov for useful discussions. This work was partially supported by the National Science Foundation under contract PHYS-90-21984 (A.M.), by the Minerva Center for Nonlinear Physics (V.L. and I.K.) and by the Rashi Foundation (G.F.).

---

- [1] I. Lifshits, S. Gredeskul and A. Pastur, *Introduction to the theory of disordered systems* (Wiley Interscience, New York 1988).
- [2] L.N. Lipatov, Sov. Phys. JETP **45**, 216 (1977).
- [3] L. D. Landau and E. M. Lifshitz, Course of Theoretical Physics VIII, E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics II, Pergamon Press, Oxford, 1980.
- [4] E. I. Kats and V. V. Lebedev, Fluctuational effects in the dynamics of liquid crystals, Springer-Verlag, N.-Y., 1993.
- [5] V. Belinicher and V. L'vov, Sov. Phys. JETP **66**, 303 (1987).
- [6] V. L'vov, Phys. Rep. **207**, 1 (1991).
- [7] H. W. Wyld, Ann. Phys. **14**, 143 (1961).
- [8] C. de Dominicis, J. Physique (Paris) **37**, c01-247 (1976).
- [9] H. Janssen, Z. Phys. B **23**, 377 (1976).
- [10] C. de Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).
- [11] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
- [12] R. H. Kraichnan, J. Fluid Mech. **64**, 737 (1974).
- [13] B. I. Shraiman and E. D. Siggia, Phys. Rev. E **49**, 2912 (1994).
- [14] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **51**, 5068 (1995).
- [15] M. Chertkov, A. Gamba, and I. Kolokolov, Phys. Lett. A **192**, 435 (1994).
- [16] V. Gurarie and A. Migdal (1995)
- [17] M.J. Giles, Phys. Fluids **7**, 2875-95 (1995).
- [18] A. Bhattacharjee, C.S. Ng and Xiaogang Wang, Phys. Rev. E **52**, 5110 (1995).
- [19] T.S. Lundgren, Phys. Fluids **25**, 2193 (1982); Phys. Fluids **A5**, 1472 (1992)